Due

# 5.4 – Differential Equations

$y'' + 5y' + 6y = 0$

2nd-order DE

**Definitions:** A **differential equation** is an equation involving unknown functions and their derivatives.

The **order** of a differential equation is the order of the highest derivative it contains.

1st-order DE

A differential equation of the form $y' = ay$ has a **general solution** of the form $y = ce^{ax}$.

If $\frac{dy}{dx} = ay$, then $\frac{dy}{y} = a\,dx \Rightarrow \int \frac{dy}{y} = \int a\,dx$

$\Rightarrow \ln|y| = ax + C_1$

$\Rightarrow y = e^{ax+C_1} \Rightarrow y = e^{C_1}e^{ax}$

$y = ce^{ax}$

A condition which specifies the value of the general solution at a point is called an **initial condition**, and the problem of solving a differential equation subject to an initial condition is called an **initial-value problem**.

If we know, for instance, that

$y(0) = P_0$, then $y = P_0 e^{ax}$.

A **constant coefficient first-order homogeneous linear system** has the form

$$\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \\
y_2' &= a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \\
\vdots & \qquad\quad \vdots \qquad \vdots \qquad\quad \vdots \\
y_n' &= a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n
\end{aligned}$$

where $y_i = f_i(x)$ are functions to be determined, and the $a_{ij}$'s are constants.

This can be written in matrix notation as

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

or $\mathbf{y}' = A\mathbf{y}$.

The **trivial solution** to the above differential equation is $y_1 = y_2 = \ldots = y_n = 0$.

If we can find a matrix $P$ that diagonalizes $A$, then we can use the diagonal matrix in solving the system. If $P$ diagonalizes $A$, then we form $\vec{y} = P\vec{u}$, where $\vec{u}$ is an unknown vector of functions. Then

$$\vec{y}\,' = A\vec{y} \Rightarrow P\vec{u}\,' = A(P\vec{u})$$
$$\Rightarrow \vec{u}\,' = P^{-1}AP\vec{u}$$

This now has the form $\vec{u}\,' = D\vec{u}$,

where $D$ is diagonal $y' = A\vec{y}$

This has the form of a system that is solved using exponentials.

Recap: Imagine an unknown vector $\vec{u}$ of functions exists. Then find $\vec{u}$ and form $\vec{y} = P\vec{u}$.

Plan: • Find eigenvalues and eigenvectors to form $P$ & $D$.
• Solve $\vec{u}' = D\vec{u}$
• $\vec{y} = P\vec{u}$

**Example:**

a. Solve the system of differential equations.

$y_1' = 2y_1 + 3y_2$

$y_2' = 2y_1 + y_2$

S.1

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \Rightarrow |\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda-2 & -3 \\ -2 & \lambda-1 \end{vmatrix} = 0$$

$$\lambda^2 - 3\lambda + 2 - 6 = 0 \Rightarrow (\lambda - 4)(\lambda+1) = 0$$

$$\lambda_1 = 4: \begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} \rightarrow \vec{P}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\lambda_2 = -1: \begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} \rightarrow \vec{P}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

S.2

$$P = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \quad \vec{u} = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}$$

S.3

$$\vec{u}' = D\vec{u} \Rightarrow \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4u_1 \\ -u_2 \end{bmatrix}$$

$$u_1' = 4u_1 \Rightarrow u_1 = c_1 e^{4x}$$

$$u_2' = -u_2 \qquad u_2 = c_2 e^{-x}$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} \quad \vec{y} = P\vec{u} \Rightarrow \vec{y} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 e^{4x} \\ c_2 e^{-x} \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 3c_1 e^{4x} + c_2 e^{-x} \\ 2c_1 e^{4x} - c_2 e^{-x} \end{bmatrix} \Rightarrow \boxed{\begin{aligned} y_1(x) &= 3c_1 e^{4x} + c_2 e^{-x} \\ y_2(x) &= 2c_1 e^{4x} - c_2 e^{-x} \end{aligned}}$$

b. Find the solution that satisfies the initial conditions $y_1(0) = 25$, $y_2(0) = 5$.

$y_1(0) = 25 \Rightarrow 25 = 3c_1 + c_2$ $\qquad c_1 = 6$

$y_2(0) = 5 \Rightarrow 5 = 2c_1 - c_2$ $\qquad c_2 = 7$

$$\boxed{\begin{aligned} y_1(x) &= 18e^{4x} + 7e^{-x} \\ y_2(x) &= 12e^{4x} - 7e^{-x} \end{aligned}}$$

2nd-order DE

**#8** Use the procedure in Exercise 7 to solve $y'' + y' - 12y = 0$.

Let $y_1 = y$ and let $y_2 = y' = y_1'$

Then $y_2' + y_2 - 12y_1 = 0$

$y_1' = y_2$

- 1st-order derivatives
- only one derivative per eqn.

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 12y_1 - y_2 \end{aligned} \Rightarrow \vec{y}\,' = \begin{bmatrix} 0 & 1 \\ 12 & -1 \end{bmatrix} \vec{y}$$

eigenvalues, eigenvectors

$$\begin{vmatrix} \lambda & -1 \\ -12 & \lambda+1 \end{vmatrix} = 0 \Rightarrow \lambda^2 + \lambda - 12 = 0$$

$\lambda_1 = -4$ $\begin{bmatrix} -4 & -1 \\ -12 & -3 \end{bmatrix} \rightarrow \vec{p}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$\lambda_2 = 3$ $\begin{bmatrix} 3 & -1 \\ -12 & 4 \end{bmatrix} \rightarrow \vec{p}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ -4 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}$

$$\vec{u}\,' = D\vec{u} \Rightarrow \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -4u_1 \\ 3u_2 \end{bmatrix}$$

$u_1' = -4u_1 \implies u_1 = c_1 e^{-4x}$

$u_2' = 3u_2 \implies u_2 = c_2 e^{3x}$

$\vec{y} = P\vec{u} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} c_1 e^{-4x} \\ c_2 e^{3x} \end{bmatrix}$

$y_2 = -4c_1 e^{-4x} + 3c_2 e^{3x}$

$y_1 = c_1 e^{-4x} + c_2 e^{3x}$

$y_2 =$ we don't care.

$y_2 = y_1' = \frac{d}{dx}(y_1) = -4c_1 e^{-4x} + 3c_2 e^{3x}$